

SMOOTHING EFFECTS FOR THE FILTRATION EQUATION WITH DIFFERENT POWERS

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ABSTRACT. We study the nonlinear diffusion equation $u_t = \Delta\phi(u)$ on general Euclidean domains, with homogeneous Neumann boundary conditions. We assume that $\phi'(u)$ is bounded from below by $|u|^{m_1-1}$ for small $|u|$ and by $|u|^{m_2-1}$ for large $|u|$, the two exponents m_1, m_2 being different and larger than one. The equality case corresponds to the well-known porous medium equation. We establish short- and long-time L^{q_0} - L^∞ smoothing estimates: similar issues have widely been investigated in the literature in the last few years, but the Neumann problem with different powers had not been addressed yet. This work extends some previous results in many directions.

1. INTRODUCTION

The present paper is devoted to the study of *smoothing* and *asymptotic* properties for solutions of the following *filtration equation* with homogeneous *Neumann* boundary conditions:

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial\phi(u)}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a continuous and increasing function vanishing at zero, Ω is a general domain of \mathbb{R}^N (not necessarily bounded or regular) and u_0 is an initial datum having suitable integrability properties that we shall specify below. Keeping in mind the widely studied case $\phi(u) = |u|^{m-1}u$ (let $m > 1$), we can also refer to (1.1) as *generalized porous medium equation*, in agreement with [49]. In fact we shall assume throughout, with the exception of Section 3, that ϕ is $C^1(\mathbb{R})$ and satisfies the following hypotheses:

$$\phi(0) = 0, \quad (1.2)$$

$$c_1 |u|^{m_1-1} \leq \phi'(u) \quad \forall u : |u| \in [0, 1], \quad (1.3)$$

$$c_2 |u|^{m_2-1} \leq \phi'(u) \quad \forall u : |u| > 1, \quad (1.4)$$

for some exponents $m_1, m_2 > 1$, with $m_1 \neq m_2$, and positive constants c_1, c_2 . In other words, if we think to (1.3)–(1.4) as equalities, we are allowing (1.1) to be like a porous medium equation with exponent m_1 where the solution is small and like a porous medium equation with another exponent m_2 where the solution is large. We shall see that m_2 is associated with short-time behaviour, whereas m_1 is associated with long-time asymptotics. It turns out that, to our purposes, the only requirements that count are bounds from *below* on ϕ' like (1.3)–(1.4), so that actually ϕ may significantly deviate from powers.

Recently, as concerns the straight porous-medium nonlinearity $\phi(u) = |u|^{m-1}u$, in [29, Theorem 3.2] it has been proved that, if Ω is bounded and regular and the spatial dimension N is greater than or equal to 3, the L^{q_0} - L^∞ smoothing effect

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0 \quad (1.5)$$

holds for all $q_0 \in [1, \infty)$ and a suitable $K > 0$. As for long-time asymptotics, such estimate can be improved depending on whether $\bar{u}_0 = 0$ or $\bar{u}_0 \neq 0$, where \bar{u}_0 is the mean value of the initial datum. In the case $\bar{u}_0 = 0$, it is shown in [29, Theorem 4.1] that

$$\|u(t)\|_\infty \leq K_1 t^{-\frac{N}{2q_0+N(m-1)}} \left(K_2 t + \|u_0\|_{q_0}^{1-m} \right)^{-\frac{2q_0}{(m-1)[2q_0+N(m-1)]}} \quad \forall t > 0 \quad (1.6)$$

holds for $q_0 \in [1, \infty)$ and suitable $K_1, K_2 > 0$, whereas in the case $\bar{u}_0 \neq 0$ [29, Theorem 4.3] establishes that

$$\|u(t) - \bar{u}_0\|_\infty \leq G e^{-\frac{m|\bar{u}_0|^{m-1}}{c_P^2} t} \quad \forall t \geq 1, \quad (1.7)$$

where G is a suitable positive constant and $C_P > 0$ is the *best constant* in the *Poincaré* inequality

$$\|f - \bar{f}\|_2 \leq C_P \|\nabla f\|_2 \quad \forall f \in H^1(\Omega), \quad \bar{f} := \frac{\int_{\Omega} f(x) dx}{|\Omega|}. \quad (1.8)$$

The above estimates were obtained by only exploiting the standard *Sobolev* inequality

$$\|f\|_{2^*} \leq C_* (\|\nabla f\|_2 + \|f\|_2) \quad \forall f \in H^1(\Omega), \quad 2^* := \frac{2N}{N-2} \quad (1.9)$$

and (1.8), the latter in order to get (1.6) and (1.7). Indeed such results could be extended to the case of *weighted* porous medium equations (or rougher domains), subject to the validity of the analogues of (1.9) and (1.8) in the corresponding framework (see [29, Section 5]). Afterwards, it was proved in [30] that (1.5)–(1.7) are still true in low dimensions, up to taking advantage of *Gagliardo-Nirenberg* inequalities. We stress that both in [29] and [30] the hypothesis $|\Omega| < \infty$ was essential.

Here we shall assume that Ω is any domain of \mathbb{R}^N satisfying the following *Gagliardo-Nirenberg-Sobolev* inequalities:

$$\|f\|_r \leq C_S (\|\nabla f\|_2 + \|f\|_2)^{\vartheta(s,r,N)} \|f\|_s^{1-\vartheta(s,r,N)} \quad \forall f \in H^1(\Omega) \cap L^s(\Omega) \quad (1.10)$$

with

$$\vartheta(s, r, N) := \frac{2N(r-s)}{r[2N-s(N-2)]}, \quad (1.11)$$

where if $N = 1$ or $N = 2$ we suppose that r, s can vary subject to

$$0 < s < r < \infty, \quad (1.12)$$

whereas in the case $N \geq 3$ we suppose they can vary subject to

$$0 < s < r \leq 2^* \quad \text{or} \quad 2^* \leq r < s < \infty. \quad (1.13)$$

The positive constant C_S is required to be bounded independently of r, s as long as the latter range in compact subsets of $(0, \infty)$. By means of Young's inequality, it is straightforward to deduce from (1.10) the validity of

$$\|f\|_r \leq C_S (\|\nabla f\|_2 + \|f\|_s)^{\vartheta(s,r,N)} \|f\|_s^{1-\vartheta(s,r,N)} \quad \forall f \in H^1(\Omega) \cap L^s(\Omega) \quad (1.14)$$

under the additional constraint $s \leq 2$ (for another constant C_S as above that we do not relabel), which will turn out to be useful in the sequel.

Inequalities (1.10) are not chosen by chance. In the seminal paper [8] it was established (in more abstract contexts actually) that the validity of (1.10) for a *single* pair (r, s) is equivalent to the validity of the whole family, i.e. of (1.10) itself for *all* r, s complying with (1.12) or (1.13), depending on the spatial dimension. In particular, if $N \geq 3$ it is readily seen that it all amounts to the Sobolev inequality (1.9). Furthermore, it is well known that (1.10) hold for regular, compactly supported functions on \mathbb{R}^N (namely in $\mathcal{D}(\mathbb{R}^N)$) with no additional L^2 norm in the right-hand side, and the latter are *equivalent* to a precise power-rate time decay for the associated *heat kernel* (see e.g. [17, Chapter 2] and Remark 5.5 below). As a consequence, they hold in the form (1.10) on all Euclidean domains having the *extension* property; more in general, at least in dimension $N \geq 3$, they hold on Euclidean domains complying with the *cone* condition. For such results, we refer the reader e.g. to the monograph [1, Chapters 4 and 5]. See also the classical, celebrated papers [27, 41] for a thorough analysis on the validity of this kind of inequalities in the Euclidean setting.

On the other hand, the Poincaré inequality (1.8) only makes sense on finite-measure domains, and it can be shown to hold on any such domain for which the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is *compact*, which is ensured (for instance) on bounded Lipschitz domains by Rellich's theorem.

Organization of the paper. In Section 2 we present our main results. Theorem 2.1 establishes smoothing effects for general L^{q_0} data, providing an estimate that is the analogue of (1.5) with $m = m_2$ and $m = m_1$ for short and long times, respectively. The corresponding proof is given in Section 4, and proceeds by means of nontrivial modifications of well-established Moser iterations (which in fact go back to [38, 39]). Theorem 2.2 then yields a better estimate (the analogue of (1.6)) under the additional assumption that the initial datum, and therefore the solution, has zero mean. The corresponding proof is given in Section 5.1. Theorem 2.3 deals with the case of data, and solutions, with nonzero mean. In view of the L^∞ smoothing effect, we can get the analogue of (1.7), in the sense that the exponential decay is the one predicted by linearization about the mean value. The proof is given in Section 5.2, and requires a little more regularity on ϕ' (see also Remark 5.7). Finally, Section 3 is devoted to providing

basic well-posedness results for problem (1.1), which however need to be treated cautiously due to the generality of our assumptions on Ω and ϕ .

Remark 1.1. Even if for simplicity we assumed $m_1 \neq m_2$ throughout, our methods do work in the straight porous-medium case (i.e. when $m_1 = m_2 = m > 1$) as well and allow us to improve on the results of [29], in the sense that we succeed in removing the hypothesis of finiteness of $|\Omega|$ (through more direct techniques, actually).

Previous results. There is a huge literature concerned with the topics addressed here. Accordingly, with no claim at all for completeness, below we quote some of the most relevant papers.

The Neumann problem for the porous medium equation, before [29, 30], had poorly been investigated in the literature. In [2] some L^∞ estimates for a similar problem with a reaction term were proved (though without establishing L^{q_0} - L^∞ regularizing effects). A pioneering breakthrough paper was [3], where the authors obtained (almost) sharp asymptotic results, as $t \rightarrow \infty$, on bounded regular domains and for initial data in $L^\infty(\Omega)$. In particular, they showed how to handle separately the zero-mean and nonzero-mean cases. In [10] smoothing effects first appeared for the Neumann problem (by means of pure functional inequalities), but they turned out not to be fully sharp as pointed out in [29].

As for long-time asymptotics, stabilization towards the mean value, in agreement with (1.6)–(1.7), is not a new phenomenon: see e.g. [22] for heat-type equations with density vanishing at infinity in one dimension, and [23] for similar results (in higher dimensions) where the degeneracy lies in the diffusion coefficients. In [21] and [32] convergence to the mean value was studied for weighted porous medium equations, by means of Poincaré-type inequalities.

Smoothing effects in the case of *Dirichlet*-type problems (or problems on the whole space) have widely been investigated, especially in the last two decades: see [48] as a comprehensive reference. We refer to [12] and [31] for L^{q_0} - L^∞ smoothing effects on *Cartan-Hadamard* manifolds, in the fast-diffusion ($m < 1$) and porous-medium case, respectively. As regards weighted porous medium equations, in [32] L^{q_0} - L^p smoothing effects (with $p \in (q_0, \infty)$) were established by only assuming a (spectral-gap) Poincaré inequality, which in general prevents L^∞ regularization. As for the *fractional* porous medium equation on Euclidean space, we quote [20] and [34], where fractional Gagliardo-Nirenberg-type (or Nash-type) inequalities were used. In [13], the same equation was considered on domains with homogeneous Dirichlet boundary conditions, and smoothing effects were proved by means of smart Green-function techniques. The p -Laplacian equation was then addressed in [28], through functional-analytic arguments involving *logarithmic* Sobolev inequalities, and in [9], showing sharp convergence to the mean value (on compact manifolds without boundary). In [11] the authors analysed doubly nonlinear equations, obtaining sharp smoothing effects still by means of a differential method that exploits logarithmic Sobolev inequalities.

Actually, smoothing estimates for Neumann problems (and general equations of p -Laplacian type) are also considered in [4], but on domains for which there hold functional inequalities which make the solution behave in a similar way to the Dirichlet case. Doubly nonlinear equations on domains narrowing at infinity are the main subject of [5], even though the geometry of the domains at hand makes again the functional setting closer to a Dirichlet-type one. For similar results on noncompact manifolds (by means of *Faber-Krahn* inequalities), see e.g. [7]. Dirichlet problems on unbounded domains, for the porous medium equation, are then analysed in [6] through *harmonic* functions that play a role in weighted Gagliardo-Nirenberg inequalities (with no additional L^2 term), so as to obtain smoothing effects with respect to the corresponding weighted norms.

An interesting alternative approach, which consists in obtaining preliminary estimates on “truncated” solutions and then pass to the limit, was deeply exploited in [43] to obtain smoothing and decay estimates for p -Laplacian-type equations and Dirichlet-type problems; further developments of such an approach were then carried out in [44] under milder conditions on p and in [45] to deal with more general equations.

We finally quote [46], where smoothing effects are obtained for *systems* of porous-medium-type equations, then generalized to the doubly nonlinear case in [47].

In the recent paper [15], a global theory of smoothing effects for nonlinear semigroups has been set up, which encompasses many of the equations discussed above. The authors proceed by means of time discretization and exploit suitable Gagliardo-Nirenberg inequalities. It is remarkable that their results hold in very abstract frameworks. Nevertheless, we point out that the problems studied here are not

included in such theory, both at the level of functional inequalities (due to the additional L^2 norm in the r.h.s. of (1.10)) and at the level of the nonlinearity we consider, which is not a single power.

Let us now turn to the filtration equation. The latter, for a rather general nonlinearity ϕ , was studied in [24] on Euclidean space with a density (existence, uniqueness and basic estimates), whereas similar issues were discussed in [25] on exterior domains. The pioneering paper [37] dealt with the one-dimensional filtration equation as concerns asymptotics via self-similar solutions, under particular conditions on ϕ . Then, in [26], the asymptotics for the same equation with finite-mass densities was investigated (proving convergence to the mean value), while in [35] (dimension one and two) the authors analysed support and blow-up properties. In general, when the density decays sufficiently fast at infinity, nontrivial well-posedness issues arise, which were deeply studied in [33].

Even *nonlocal* versions of the filtration equation have recently been investigated. In [14] the authors address the Dirichlet problem for a very general equation which covers both the local and the nonlocal case, under suitable assumptions on the *Green* function associated with the operator considered. In particular, up to slightly stronger requirements on ϕ (see Section 2 there), they obtain smoothing effects (see Corollary 6.3 there) analogous to those discussed in Remark 5.5: to the best of our knowledge, this is the very first paper dealing with a function ϕ that is allowed to have two different power-type behaviours at zero and at infinity. As for the problem on Euclidean space, in [50] fine regularity results have been shown in the case where the nonlocal operator is the standard fractional Laplacian, whereas in [19] similar properties have been investigated for operators with rougher kernels.

For a wide dissertation on filtration equations, we also refer the reader to [16], even though there the analysis is mostly concerned with regularity properties and estimates for nonnegative local solutions, solutions on the whole Euclidean space or solutions of the Dirichlet problem on regular domains, especially when ϕ is trapped between two powers at infinity.

2. STATEMENTS OF THE MAIN RESULTS

We describe here our results concerning smoothing and asymptotic estimates for solutions of (1.1), under suitable hypotheses on Ω that only involve the validity of functional inequalities, as discussed in the Introduction. A precise meaning to the concept of “solution” will be given in Section 3, see in particular Remark 3.7 there.

Theorem 2.1 (Smoothing). *Let $\Omega \subset \mathbb{R}^N$ be a domain which satisfies the Gagliardo-Nirenberg-Sobolev inequalities (1.10). Let u be the solution of (1.1) corresponding to an initial datum $u_0 \in L^1(\Omega) \cap L^{q_0}(\Omega)$ with $q_0 \in [1, \infty)$, where $\phi \in C^1(\mathbb{R})$ is any nonlinearity complying with (1.2)–(1.4). Then the following smoothing estimate holds:*

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) & \forall t \in \left(0, \|u_0\|_{q_0}^{\frac{2q_0}{N}}\right), \\ K \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} + \|u_0\|_{q_0} \right) & \forall t \geq \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \end{cases} \quad (2.1)$$

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, C_S, N$.

In the case of domains with finite measure, upon assuming the validity of the Poincaré inequality, the above result can be improved, especially as concerns long-time asymptotics. In this regard, it is crucial to treat separately data (and therefore solutions) with zero and nonzero mean.

Theorem 2.2 (Smoothing and asymptotics, $\bar{u}_0 = 0$). *Let the hypotheses of Theorem 2.1 be fulfilled. Suppose moreover that Ω is of finite measure and such that the Poincaré inequality (1.8) holds. Let $\bar{u}_0 = 0$. Then the following estimates hold:*

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} & \forall t \in \left(0, \|u_0\|_{q_0}^{\frac{2q_0}{N}}\right), \\ K t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} & \forall t \in \left[\|u_0\|_{q_0}^{\frac{2q_0}{N}}, 1\right], \\ K t^{-\frac{N}{2q_0+N(m_1-1)}} (t + \|u_0\|_{q_0}^{1-m_1})^{-\frac{2q_0}{(m_1-1)[2q_0+N(m_1-1)]}} & \forall t > 1, \end{cases} \quad (2.2)$$

for all $u_0 : \|u_0\|_{q_0} \leq 1$,

and

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{N}{2q_0+N(m_2-1)}} (t + \|u_0\|_{q_0}^{1-m_2})^{-\frac{2q_0}{(m_2-1)[2q_0+N(m_2-1)]}} & \forall t \in (0, 1), \\ K t^{-\frac{1}{m_1-1}} & \forall t > 1, \end{cases} \quad (2.3)$$

for all $u_0 : \|u_0\|_{q_0} > 1$,

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, \Omega$.

Theorem 2.3 (Asymptotics, $\bar{u}_0 \neq 0$). *Let the hypotheses of Theorem 2.1 be fulfilled, with the additional assumption $\phi \in C^2(\mathbb{R} \setminus \{0\})$. Suppose moreover that Ω is of finite measure and such that the Poincaré inequality (1.8) holds. Let $\bar{u}_0 \neq 0$. Then the following estimate holds:*

$$\|u(t) - \bar{u}_0\|_\infty \leq G e^{-\frac{\phi'(\bar{u}_0)}{C_P^2} t} \|u_0 - \bar{u}_0\|_1 \quad \forall t \geq 1, \quad (2.4)$$

where C_P is the best constant in (1.8) and G is a positive constant depending only on $\|u_0\|_1, |\bar{u}_0|, \phi, \Omega$, which can be taken to be increasing w.r.t. $\|u_0\|_1$ and locally bounded w.r.t. $|\bar{u}_0| > 0$.

Theorems 2.1, 2.2 and 2.3 will be proved in Sections 4, 5.1 and 5.2, respectively.

Remark 2.4. The strategies of proof we employ, based on Moser iterations and semigroup arguments, are quite general, and in particular can be straightforwardly adapted to deal with analogous problems on Riemannian manifolds, in the spirit e.g. of [10, 12, 31], or problems with weights, in the spirit e.g. of [29, 32, 30]. The only essential requirement we need is the validity of functional inequalities of the type of (1.10) or (1.8) (in the case of Riemannian manifolds, see the classical reference [36]). However, in order not to divert the discussion from the main topics, we preferred to work in the standard framework of Euclidean domains.

3. WELL-POSEDNESS AND BASIC PROPERTIES

In this section we deal with existence and uniqueness issues for solutions of problem (1.1) (and related properties), so as to clarify what we mean by “solution” in the results stated above. Even though we are mainly interested in functions $\phi \in C^1(\mathbb{R})$ complying with (1.2)–(1.4), in the sequel we shall allow for more general nonlinearities, that is we shall only make the following assumptions (see e.g. [49, Section 5.2]):

$$\phi : \mathbb{R} \mapsto \mathbb{R} \text{ is continuous and strictly increasing, with } \lim_{u \rightarrow \pm\infty} \phi(u) = \pm\infty, \quad \phi(0) = 0. \quad (3.1)$$

Similar remarks hold for Ω : we only suppose that it is a general domain of \mathbb{R}^N , regardless of boundedness, regularity or the validity of global functional inequalities like (1.10) and (1.8). However, in the cases where the assumptions of Theorem 2.1 are fulfilled, we can somewhat improve on the well-posedness theory outlined here (see Remark 3.7 below in this regard).

Let us start off by providing an appropriate definition of weak solution.

Definition 3.1 (Weak solutions). *Given $u_0 \in L^1(\Omega)$, a measurable function u is a weak solution of the Neumann problem (1.1) if, for all $T > 0$,*

$$u \in L^1(\Omega \times (0, T)), \quad \phi(u) \in L^1_{\text{loc}}(\Omega \times (0, T)), \quad \nabla \phi(u) \in L^1((0, T); [L^2(\Omega)]^N)$$

and

$$\int_0^T \int_\Omega u(x, t) \eta_t(x, t) dx dt = - \int_\Omega u_0(x) \eta(x, 0) dx + \int_0^T \int_\Omega \nabla \phi(u)(x, t) \cdot \nabla \eta(x, t) dx dt$$

for all $\eta \in W^{1,\infty}((0, T); L^\infty(\Omega))$ with $\nabla \eta \in L^\infty((0, T); [L^2(\Omega)]^N)$, such that $\eta(\cdot, T) \equiv 0$.

Remark 3.2. As they reader may notice, our definition of weak solution slightly differs from [49, Definition 11.2]. The point is that, as just recalled above, we aim at considering general domains Ω : in particular, density of functions that are regular up to the boundary need not hold e.g. in $H^1(\Omega)$. For this reason we do not assume $\eta \in C^1(\bar{\Omega} \times [0, T])$ but we ask that η belongs to a larger “dual” space, namely $\nabla \eta \in [L^2(\Omega)]^N$. As a reference for similar questions, see also [32, Sections 2, 3].

As a direct consequence Definition 3.1, all weak solutions enjoy an important property.

Proposition 3.3 (Mass conservation). *Let u be any weak solution of (1.1). Then*

$$\int_\Omega u(x, t) dx = \int_\Omega u_0(x) dx \quad \text{for a.e. } t > 0.$$

Proof. One can proceed exactly as in the proof of [32, Proposition 9], that is by using the (constant-in-space) test function $\eta = \chi_{[0,t]}$ up to approximations. In the light of Definition 3.1, the finiteness of $|\Omega|$ is in fact not necessary. \square

Before stating a key existence result (i.e. the analogue of [49, Theorem 11.2]), we need to introduce the primitive function of ϕ , namely $\psi(u) := \int_0^u \phi(v) dv$. Note that $\psi(0) = 0$ and, by virtue of (3.1), ψ is $C^1(\mathbb{R})$, positive in $\mathbb{R} \setminus \{0\}$, strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$, with $\lim_{u \rightarrow \pm\infty} \psi(u) = +\infty$.

Theorem 3.4 (Existence and estimates). *Let $u_0 \in L^1(\Omega)$ with $\psi(u_0) \in L^1(\Omega)$. Then there exists a weak solution u of (1.1), which enjoys the following properties.*

- Energy inequality: u satisfies

$$\int_0^T \int_{\Omega} |\nabla \phi(u)(x, t)|^2 dx dt + \int_{\Omega} \psi(u(x, T)) dx \leq \int_{\Omega} \psi(u_0(x)) dx \quad \text{for a.e. } T > 0 \quad (3.2)$$

and is referred to as weak energy solution.

- Approximation: u is obtained as a limit of classical solutions of suitable non-degenerate parabolic problems.
- L^1 -contractivity and comparison: if v is another weak energy solution corresponding to some $v_0 \in L^1(\Omega)$ with $\psi(v_0) \in L^1(\Omega)$, then

$$\|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1 \quad \text{for a.e. } t > 0. \quad (3.3)$$

Moreover, if $v_0 \leq u_0$ a.e. in Ω then $u \leq v$ a.e. $\Omega \times \mathbb{R}^+$.

- Non-expansivity of the norms: if in addition $u_0 \in L^\infty(\Omega)$, there holds

$$\|u(t)\|_p \leq \|u_0\|_p \quad \forall p \in [1, \infty], \text{ for a.e. } t > 0. \quad (3.4)$$

Proof. The procedure for constructing a weak energy solution is by now quite standard, though elaborate. It is not of our concern to illustrate it here: for the reader's convenience, we refer to [49, Sections 5.5, 11.2], [32, Section 3] and [40, Section 1.3.2] for accurate step-by-step proofs (in various contexts).

We recall that the main idea is to begin with considering (regular) data $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$, solve the problem upon replacing ϕ with a suitable sequence of regular, non-degenerate nonlinearities ϕ_n approximating ϕ and then pass to the limit as $n \rightarrow \infty$. At this level Ω is supposed to be also bounded and regular. Afterwards, one first removes the assumptions on the domain by taking a sequence of regular domains such that $\bar{\Omega}_n \Subset \Omega$ which eventually covers the whole Ω , and finally removes the assumptions on the initial datum by picking a sequence of regular initial data $u_{0n} \in L^1(\Omega) \cap L^\infty(\Omega)$ such that $u_{0n} \rightarrow u_0$ and $\psi(u_{0n}) \rightarrow \psi(u_0)$ in $L^1(\Omega)$. In most of these steps, estimates (3.2), (3.3) and (3.4) (to be understood on the approximating sequences of solutions), plus additional estimates that we do not mention, are crucially exploited.

Let us just discuss a subtle question in the last passage to the limit, where we cannot use (3.4). That is, in order to show that $\nabla \phi(u_n)$ converges (weakly) to $\nabla \phi(u)$ in $L^2((0, T); [L^2(\Omega)]^N)$, it is essential to establish that $\phi(u_n)$ converges to $\phi(u)$ at least (weakly) in $L^1_{\text{loc}}(\Omega \times (0, T))$. To this aim, first of all note that by means of the local Poincaré inequality, for all bounded, regular $\bar{\Omega}_0 \Subset \Omega$ there holds

$$\|\phi(u_n)(t) - \overline{\phi(u_n)(t)}\|_{L^2(\Omega_0)} \leq C_P(\Omega_0) \|\nabla \phi(u_n)(t)\|_{L^2(\Omega_0)} \quad \text{for a.e. } t \in (0, T), \quad (3.5)$$

where for simplicity we still denote by \bar{f} we the mean value of $f \in L^1(\Omega_0)$ in Ω_0 . Thanks to the energy inequality we know, in particular, that

$$|\{x \in \Omega : |u_n(x, t)| > c\}| \leq \frac{b}{\min\{\psi(c), \psi(-c)\}}, \quad b := \limsup_{n \rightarrow \infty} \int_{\Omega} \psi(u_{0n}(x)) dx < \infty, \quad (3.6)$$

for all $c > 0$, where we used the fact that ψ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$, with $\psi(0) = 0$. Since $\lim_{u \rightarrow \pm\infty} \psi(u) = +\infty$, upon picking c large enough (independently of n), from (3.6) we infer that there exists $E_n \subset \Omega_0$ such that

$$|u_n(x, t)| \leq c \quad \text{for a.e. } x \in E_n, \quad |E_n| \geq \frac{1}{2} |\Omega_0|. \quad (3.7)$$

As ϕ is increasing, by combining (3.5) with (3.7) we obtain

$$|\overline{\phi(u_n)(t)}| \leq \sqrt{2}M + C_P(\Omega_0) \sqrt{2|\Omega_0|^{-1}} \|\nabla\phi(u_n)(t)\|_{L^2(\Omega_0)}, \quad M := \max\{\phi(c), -\phi(-c)\}. \quad (3.8)$$

By exploiting (3.8) and again (3.5), we therefore end up with

$$\|\phi(u_n)(t)\|_{L^2(\Omega_0)} \leq \sqrt{2|\Omega_0|} M + (\sqrt{2} + 1) C_P(\Omega_0) \|\nabla\phi(u_n)(t)\|_{L^2(\Omega_0)}. \quad (3.9)$$

Hence, if we square (3.9), integrate in $(0, T)$ and use the energy inequality, we finally deduce that $\phi(u_n)$ is bounded in $L^2(\Omega_0 \times [0, T])$, which is enough to our purposes (pointwise convergence to $\phi(u)$ is already ensured e.g. by the L^1 -contractivity).

We stress that in the plain porous-medium or fast-diffusion cases, namely $\phi(u) = u^m$ for some $m > 0$, the above convergence is a simple consequence of the boundedness of u_n in $L^{m+1}(\Omega \times (0, T))$. \square

For initial data that only belong to $L^1(\Omega)$, in general we cannot guarantee the existence of a weak solution. Nevertheless, in such case there is still a natural way to define what one means by “solution”.

Theorem 3.5 (Limit solutions). *There exists a well-defined map acting from $L^1(\Omega)$ to $L^\infty(\mathbb{R}^+; L^1(\Omega))$ which associates with each $u_0 \in L^1(\Omega)$ the limit u in $L^\infty(\mathbb{R}^+; L^1(\Omega))$ of (any) sequence of weak energy solutions u_n corresponding to initial data $u_{0n} \in L^1(\Omega)$, with $\psi(u_{0n}) \in L^1(\Omega)$, such that $u_{0n} \rightarrow u_0$ in $L^1(\Omega)$. The L^1 -contraction and the comparison properties are preserved at the limit.*

Proof. This is a standard fact, see e.g. [49, Theorems 6.2 and 11.3]: it all amounts to showing that the limit does not depend on the particular sequence of initial data (and related weak energy solutions according to Theorem 3.4) chosen. \square

Uniqueness of weak (energy) solutions for problem (1.1) is due to a classical trick, which goes back to Oleĭnik [42]. However, we emphasize that in the forthcoming theorem we do not require further integrability or boundedness properties on u or $\phi(u)$, in contrast with known results available in the literature: see for instance [49, Theorems 5.3 and 11.1] and [32, Propositions 6 and 8] in the local case or [19, Theorem 1.1] and [18, Theorem 2.4 and Corollaries 2.7–2.9] in the case of generalized nonlocal filtration equations with rough kernels. This can be done by means of an elementary truncation argument.

Theorem 3.6 (Oleĭnik’s uniqueness). *There exists at most one weak solution u of (1.1) such that*

$$\nabla\phi(u) \in L^2(\Omega \times (0, T)) \quad \forall T > 0. \quad (3.10)$$

Proof. The basic idea consists in plugging

$$\eta(x, t) = \int_t^T [\phi(u(x, s)) - \phi(v(x, s))] ds \quad (3.11)$$

in the weak formulation satisfied by the difference between u and v (the latter being any possibly different solution fulfilling (3.10) with u replaced by v) and perform the same computations as in the proof of [49, Theorem 5.3] (see also [32, Propositions 6 and 8]). We only need to justify the use of (3.11) as a test function. To this aim, let us set

$$\eta_n := \int_t^T [\phi(u_n(x, s)) - \phi(v_n(x, s))] ds,$$

where $u_n := -n \vee (n \wedge u)$ and $v_n := -n \vee (n \wedge v)$; note that each η_n is indeed an admissible test function according to Definition 3.1. Upon picking η_n in place of η we obtain the key identity

$$\begin{aligned} & \int_0^T \int_\Omega (u(x, t) - v(x, t)) [\phi(u_n(x, t)) - \phi(v_n(x, t))] dx dt \\ & + \int_0^T \int_\Omega \nabla[\phi(u) - \phi(v)](x, t) \cdot \left(\int_t^T \nabla[\phi(u_n) - \phi(v_n)](x, s) ds \right) dx dt = 0. \end{aligned}$$

Since ϕ is increasing we can pass to the limit by monotone convergence in the first integral, while in the second integral we can exploit dominated convergence in view of (3.10). The identity $u \equiv v$ then follows as in the proof of [49, Theorem 5.3]. \square

Remark 3.7. When referring to the “solution” of problem (1.1) in Theorems 2.1, 2.2 and 2.3, we shall mean the weak energy solution constructed in Theorem 3.4 if $u_0 \in L^1(\Omega)$ with $\psi(u_0) \in L^1(\Omega)$ (note that initial data belonging to $L^1(\Omega) \cap L^\infty(\Omega)$ are always included in this category), which by virtue of Theorem 3.6 is the *unique* weak solution complying with (3.10). In the general case $u_0 \in L^1(\Omega)$ we shall just mean the limit solution provided by Theorem 3.5. We stress that, in agreement with the latter results, such solutions can always be thought as limits (after several approximations) of regular solutions to suitable non-degenerate parabolic problems, a fact that we shall exploit in order to justify some of the computations performed in Sections 4 and 5.

Weak energy solutions are in fact continuous curves in $L^1(\Omega)$, i.e. $u \in C([0, \infty); L^1(\Omega))$. This is due to an alternative method for constructing such solutions by means of *time discretization*, which takes advantage of the celebrated Crandall-Liggett theorem: see [49, Chapter 10]. The L^1 -continuity is then trivially inherited by limit solutions: for that reason, and in order to lighten the reading, throughout Sections 2, 4, 5 we wrote “ $\forall t$ ” in place of “for a.e. t ”.

Actually, by exploiting the latter properties, we can infer uniqueness in a wider class of solutions, namely that of functions which are weak energy solutions for all *positive* times (i.e. with the time origin shifted to ε , for all $\varepsilon > 0$) and belong to $C([0, \infty); L^1(\Omega))$. The corresponding argument is analogous to the one used in the proof of [49, Theorem 6.12]. We stress that under assumptions (1.2)–(1.4), thanks to Theorem 2.1, *a posteriori* limit solutions are included in such class.

4. SHORT-TIME ESTIMATES: PROOFS

We begin with the analysis of the case $m_1 > m_2$, where proofs are more involved since it is not possible to deduce from (1.3)–(1.4) a global lower bound on $\phi'(u)$ that depends on a *single* power.

Lemma 4.1. *Let the hypotheses of Theorem 2.1 be fulfilled, with the additional assumptions $m_1 > m_2$, $u_0 \in L^{q_0}(\Omega) \cap L^\infty(\Omega)$ and $q_0 > 1$. Let*

$$t^* := \sup \{t \geq 0 : \|u(t)\|_\infty > 1\} \quad (4.1)$$

and suppose that $t^ > 0$. Then there holds the smoothing estimate*

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0 + N(m_2 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_2 - 1)}} + \|u_0\|_{q_0} \right) \quad \forall t \in (0, t^*), \quad (4.2)$$

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, C_S, N, q_0$.

Proof. Let $M := \|u_0\|_\infty$. Note that the non-expansivity of the norms (Theorem 3.4) and the assumption $t^* > 0$ imply $\|u_0\|_\infty > 1$. Still by means of the non-expansivity of the norms, we know that $\|u(t)\|_\infty \leq M$ for all $t > 0$, so that u takes on values in the interval $[-M, M]$. As a consequence, the bounds (1.3)–(1.4) ensure that

$$\frac{c}{M^{m_1 - m_2}} |u(x, t)|^{m_1 - 1} \leq \phi'(u(x, t)) \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad (4.3)$$

where $c := c_1 \wedge c_2$. Thanks to (4.3), which allows us to partially resort to the single-power case (*i.e.* to the standard porous medium equation), first of all we can set up a Moser-iteration scheme in the same spirit as [29, 30], and then by means of another iteration we shall remove the dependence on M .

Given $t > 0$, let us consider the sequence of time steps $t_k = (1 - 2^{-k})t$, for all $k \in \mathbb{N}$. Clearly, $t_0 = 0$ and $t_\infty = t$. Also, let p_k be an increasing sequence of positive numbers such that $p_0 = q_0$ and $p_\infty = \infty$, which we shall explicitly define later. By multiplying the differential equation in (1.1) by $u^{p_k - 1}$, integrating by parts in $\Omega \times (t_k, t_{k+1})$ and using (4.3), we obtain:

$$\begin{aligned} & \frac{4c p_k (p_k - 1)}{M^{m_1 - m_2} (m_1 + p_k - 1)^2} \int_{t_k}^{t_{k+1}} \int_\Omega \left| \nabla \left(u^{\frac{m_1 + p_k - 1}{2}} \right) (x, t) \right|^2 dx dt \\ & \leq p_k (p_k - 1) \int_{t_k}^{t_{k+1}} \int_\Omega |u(x, t)|^{p_k - 2} \phi'(u(x, t)) |\nabla u(x, t)|^2 dx dt \\ & = \|u(t_k)\|_{p_k}^{p_k} - \|u(t_{k+1})\|_{p_k}^{p_k} \\ & \leq \|u(t_k)\|_{p_k}^{p_k}. \end{aligned} \quad (4.4)$$

We point out that *a priori* u may not possess enough regularity to justify rigorously the above computation: nevertheless, this issue can be overcome by means of suitable approximation schemes, see e.g. [32, Proof of Lemma 3.3 and Remark 2] and the first part of Remark 3.7 (similar comments apply to the

computations carried out below). In order to handle the left-hand side of inequality (4.4), it is convenient to apply the Gagliardo-Nirenberg-Sobolev inequality in the form (1.14) to the function $f = u^{(m_1+p_k-1)/2}$:

$$\begin{aligned} & \frac{2c p_k (p_k - 1)}{M^{m_1-m_2} C_S^{\frac{2}{\theta}} (m_1 + p_k - 1)^2} \int_{t_k}^{t_{k+1}} \frac{\|u(t)\|^{\frac{m_1+p_k-1}{\theta}}}{\|u(t)\|^{\frac{(1-\theta)(m_1+p_k-1)}{s(m_1+p_k-1)}}} dt \\ & \leq \|u(t_k)\|_{p_k}^{p_k} + \frac{4c p_k (p_k - 1)}{M^{m_1-m_2} (m_1 + p_k - 1)^2} \int_{t_k}^{t_{k+1}} \|u(t)\|^{\frac{m_1+p_k-1}{s(m_1+p_k-1)}} dt. \end{aligned} \quad (4.5)$$

By choosing

$$s = \frac{2p_k}{m_1 + p_k - 1} \quad \text{and} \quad r = 2 + \frac{2s}{N} = 2 \frac{(N+2)p_k + N(m_1 - 1)}{N(m_1 + p_k - 1)}$$

(note that $s < 2$ and (1.12)–(1.13) are always fulfilled), recalling (1.11), the definition of t_k and using the non-expansivity of the norms, from (4.5) we infer

$$\begin{aligned} & \frac{c p_k (p_k - 1) t}{M^{m_1-m_2} C_S^{\frac{2}{\theta}} (m_1 + p_k - 1)^2 2^k} \times \frac{\|u(t_{k+1})\|_{p_{k+1}}^{p_{k+1}}}{\|u(t_k)\|_{p_k}^{\frac{2p_k}{N}}} \\ & \leq \|u(t_k)\|_{p_k}^{p_k} + \frac{2c p_k (p_k - 1) t}{M^{m_1-m_2} (m_1 + p_k - 1)^2 2^k} \|u(t_k)\|_{p_k}^{m_1+p_k-1}, \end{aligned} \quad (4.6)$$

where p_{k+1} is defined recursively by

$$p_{k+1} = \frac{N+2}{N} p_k + m_1 - 1,$$

or equivalently

$$p_k = \left[q_0 + \frac{N(m_1 - 1)}{2} \right] \left(\frac{N+2}{2} \right)^k - \frac{N(m_1 - 1)}{2}. \quad (4.7)$$

From here on we shall denote by D a generic positive constant that depends only on m_1, m_2, c, C_S, N, q_0 . Hence, upon observing that $C_S^{1/\theta}$ can be bounded independently of s and r chosen as above, estimate (4.6) reads

$$\|u(t_{k+1})\|_{p_{k+1}}^{p_{k+1}} \leq D \left(\frac{2^k M^{m_1-m_2}}{t} \|u(t_k)\|_{p_k}^{\frac{N+2}{N} p_k} + \|u(t_k)\|_{p_k}^{\frac{N+2}{N} p_k + m_1 - 1} \right). \quad (4.8)$$

By combining the non-expansivity of the norms, the monotonicity of p_k , interpolation and Young's inequality, we obtain:

$$\|u(t_k)\|_{p_k} \leq \|u_0\|_{\infty} + \|u_0\|_{q_0} = M + \|u_0\|_{q_0},$$

so that (4.8) implies

$$\|u(t_{k+1})\|_{p_{k+1}} \leq D^{\frac{k+1}{p_{k+1}}} \left(\frac{M^{m_1-m_2}}{t} + M^{m_1-1} + \|u_0\|_{q_0}^{m_1-1} \right)^{\frac{1}{p_{k+1}}} \|u(t_k)\|_{p_k}^{\frac{N+2}{N} \frac{p_k}{p_{k+1}}}. \quad (4.9)$$

The iteration of (4.9) and again the non-expansivity of the norms yield

$$\|u(t)\|_{p_{k+1}} \leq D^{\frac{\sum_{h=1}^{k+1} h \left(\frac{N+2}{N} \right)^{k+1-h}}{p_{k+1}}} \left(\frac{M^{m_1-m_2}}{t} + M^{m_1-1} + \|u_0\|_{q_0}^{m_1-1} \right)^{\frac{\sum_{h=0}^k \left(\frac{N+2}{N} \right)^h}{p_{k+1}}} \|u_0\|_{q_0}^{\left(\frac{N+2}{N} \right)^{k+1} \frac{q_0}{p_{k+1}}}.$$

By letting $k \rightarrow \infty$, in view of (4.7) we end up with

$$\|u(t)\|_{\infty} \leq D \left(\frac{M^{m_1-m_2}}{t} + M^{m_1-1} + \|u_0\|_{q_0}^{m_1-1} \right)^{\frac{N}{2q_0 + N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1-1)}},$$

whence

$$\|u(t)\|_{\infty} \leq D \left(\frac{M^{\frac{m_1-m_2}{m_1-1}}}{t^{\frac{1}{m_1-1}}} + M + \|u_0\|_{q_0} \right)^{\frac{N(m_1-1)}{2q_0 + N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1-1)}}. \quad (4.10)$$

Thanks to Young's inequality

$$A^{\theta} B^{1-\theta} \leq \varepsilon \theta A + \varepsilon^{-\frac{\theta}{1-\theta}} (1-\theta) B \quad \forall A, B, \varepsilon > 0, \quad \theta := \frac{N(m_1-1)}{2q_0 + N(m_1-1)},$$

from (4.10) we infer

$$\|u(t)\|_\infty \leq D \left[\frac{M^{\frac{m_1-m_2}{m_1-1} \theta}}{t^{\frac{\theta}{m_1-1}}} \|u_0\|_{q_0}^{1-\theta} + \varepsilon \theta M + \left[\varepsilon \theta + \varepsilon^{-\frac{\theta}{1-\theta}} (1-\theta) \right] \|u_0\|_{q_0} \right]. \quad (4.11)$$

Now let us pick

$$\varepsilon = \left(D \theta 2^{\frac{1+\theta}{m_1-1}} \right)^{-1},$$

so that (4.11) can be rewritten as

$$\|u(t)\|_\infty \leq \frac{1}{2^{\frac{1+\theta}{m_1-1}}} M + D \frac{M^{\frac{m_1-m_2}{m_1-1} \theta}}{t^{\frac{\theta}{m_1-1}}} \|u_0\|_{q_0}^{1-\theta} + D \|u_0\|_{q_0} \quad (4.12)$$

for another positive constant D which, as usual, we do not relabel. Our goal is to (partially) remove the dependence of the right-hand side of (4.12) on M : to this end, first of all we exploit a classical $t/2$ -shift argument (see e.g. [29, Proof of Theorem 3.2]) combined with the non-expansivity of the norms, which yields

$$\|u(t/2^k)\|_\infty \leq \frac{\|u(t/2^{k+1})\|_\infty}{2^{\frac{1+\theta}{m_1-1}}} + 2^{\frac{\theta(k+1)}{m_1-1}} D \frac{M^{\frac{m_1-m_2}{m_1-1} \theta}}{t^{\frac{\theta}{m_1-1}}} \|u_0\|_{q_0}^{1-\theta} + D \|u_0\|_{q_0} \quad \forall k \in \mathbb{N}. \quad (4.13)$$

By iterating (4.13) we therefore obtain

$$\|u(t)\|_\infty \leq \frac{M}{2^{\frac{(1+\theta)(\ell+1)}{m_1-1}}} + 2^{\frac{\theta \ell}{m_1-1}} D \frac{M^{\frac{m_1-m_2}{m_1-1} \theta}}{t^{\frac{\theta}{m_1-1}}} \|u_0\|_{q_0}^{1-\theta} \sum_{k=0}^{\ell} 2^{-\frac{k}{m_1-1}} + D \|u_0\|_{q_0} \sum_{k=0}^{\ell} 2^{-\frac{(1+\theta)k}{m_1-1}}$$

for all $\ell \in \mathbb{N}$, whence, upon taking limits as $\ell \rightarrow \infty$,

$$\|u(t)\|_\infty \leq D \left(\frac{M^{\frac{m_1-m_2}{m_1-1} \theta}}{t^{\frac{\theta}{m_1-1}}} \|u_0\|_{q_0}^{1-\theta} + \|u_0\|_{q_0} \right). \quad (4.14)$$

In order to remove definitively the dependence of the right-hand side of (4.14) on M we can argue in a similar way as above to get, by means of Young's inequality,

$$\|u(t)\|_\infty \leq D \left(\varepsilon \theta_* M + \varepsilon^{-\frac{\theta_*}{1-\theta_*}} (1-\theta_*) \frac{\|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}}}{t^{\frac{N}{2q_0+N(m_2-1)}}} + \|u_0\|_{q_0} \right), \quad (4.15)$$

where we set

$$\theta_* := \frac{m_1-m_2}{m_1-1} \theta = \frac{N(m_1-m_2)}{2q_0+N(m_1-1)}.$$

By choosing

$$\varepsilon = \left(D \theta_* 2^{\frac{2q_0+2N(m_2-1)}{(m_2-1)[2q_0+N(m_2-1)]}} \right)^{-1}$$

and exploiting again a $t/2$ -shift argument, from (4.15) we infer

$$\|u(t/2^k)\|_\infty \leq \frac{\|u(t/2^{k+1})\|_\infty}{2^{\frac{2q_0+2N(m_2-1)}{(m_2-1)[2q_0+N(m_2-1)]}}} + 2^{\frac{N(k+1)}{2q_0+N(m_2-1)}} D \frac{\|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}}}{t^{\frac{N}{2q_0+N(m_2-1)}}} + D \|u_0\|_{q_0} \quad \forall k \in \mathbb{N}. \quad (4.16)$$

It is apparent that (4.16) is of the same type as (4.13): by carrying out an analogous iteration, we therefore end up with (4.2). \square

We now extend the above result to $q_0 = 1$.

Corollary 4.2. *Let the hypotheses of Theorem 2.1 be fulfilled, with the additional assumptions $m_1 > m_2$ and $u_0 \in L^\infty(\Omega)$. Let $t^* > 0$ be defined by (4.1). Then there holds the smoothing estimate*

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) \quad \forall t \in (0, t^*), \quad (4.17)$$

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, C_S, N$.

Proof. First of all, let us show that (4.2) holds down to $q_0 = 1$: note that we cannot simply let $q_0 \rightarrow 1^+$ since the multiplicative constant blows up (this is due to (4.4) evaluated at $p_k = q_0 = 1$). Hence, to begin with, fix any $q_0 > 1$ and plug the interpolation inequality $\|u_0\|_{q_0} \leq \|u_0\|_\infty^{1-1/q_0} \|u_0\|_1^{1/q_0}$ in (4.2) to get

$$\|u(t)\|_\infty \leq K \|u_0\|_\infty^{\frac{2(q_0-1)}{2q_0+N(m_2-1)}} \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2q_0+N(m_2-1)}} + \|u_0\|_\infty^{\frac{N(m_2-1)(q_0-1)}{q_0[2q_0+N(m_2-1)]}} \|u_0\|_1^{\frac{1}{q_0}} \right). \quad (4.18)$$

By means of the usual $t/2$ -shift argument and the non-expansivity of the norms, from (4.18) we deduce

$$\begin{aligned} \|u(t/2^k)\|_\infty &\leq 2^{\frac{N(k+1)}{2q_0+N(m_2-1)}} K \|u(t/2^{k+1})\|_\infty^{\frac{2(q_0-1)}{2q_0+N(m_2-1)}} \\ &\quad \times \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2q_0+N(m_2-1)}} + \|u_0\|_\infty^{\frac{N(m_2-1)(q_0-1)}{q_0[2q_0+N(m_2-1)]}} \|u_0\|_1^{\frac{1}{q_0}} \right) \end{aligned} \quad (4.19)$$

for all $k \in \mathbb{N}$. A straightforward iteration of (4.19) yields

$$\|u(t)\|_\infty \leq D \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2q_0+N(m_2-1)}} + \|u_0\|_\infty^{\frac{N(m_2-1)(q_0-1)}{q_0[2q_0+N(m_2-1)]}} \|u_0\|_1^{\frac{1}{q_0}} \right)^{\frac{2q_0+N(m_2-1)}{2+N(m_2-1)}},$$

that is

$$\|u(t)\|_\infty \leq D \left(t^{-\frac{N}{2+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2+N(m_2-1)}} + \|u_0\|_\infty^{\frac{N(m_2-1)(q_0-1)}{q_0[2+N(m_2-1)]}} \|u_0\|_1^{\frac{2q_0+N(m_2-1)}{q_0[2+N(m_2-1)]}} \right) \quad (4.20)$$

up to relabelling D , which denotes again a generic positive constant that depends only on the quantities $m_1, m_2, c_1, c_2, C_S, N, q_0$. In order to remove the dependence of the right-hand side of (4.20) on $\|u_0\|_\infty$, one can proceed as in the proof of Lemma 4.1: by combining Young's inequality with a $t/2$ -shift argument, from (4.20) we infer

$$\|u(t/2^k)\|_\infty \leq \frac{\|u(t/2^{k+1})\|_\infty}{2^{\frac{2+2N(m_2-1)}{(m_2-1)[2+N(m_2-1)]}}} + 2^{\frac{N(k+1)}{2+N(m_2-1)}} D t^{-\frac{N}{2+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2+N(m_2-1)}} + D \|u_0\|_1. \quad (4.21)$$

The recursive relation (4.21) is of the type of (4.13), so that by reasoning as above we end up with

$$\|u(t)\|_\infty \leq D \left(t^{-\frac{N}{2+N(m_2-1)}} \|u_0\|_1^{\frac{2}{2+N(m_2-1)}} + \|u_0\|_1 \right),$$

which is precisely (4.2) in the case $q_0 = 1$.

Finally, we are left with proving that (4.17) holds, namely (4.2) with a constant K that can be taken to be independent of q_0 . In fact it is straightforward to verify that all of the estimates provided along the proof of Lemma 4.1 depend continuously on $q_0 \in (1, \infty)$ and are stable as $q_0 \rightarrow \infty$. On the other hand, the argument we used here to extend the estimate to $q_0 = 1$ can be performed analogously for any $q'_0 \in (1, q_0)$, thus providing the same estimate as (4.2) with a constant K that stays bounded as $q'_0 \rightarrow 1$. \square

In order to obtain estimate (2.1) and therefore complete the proof of Theorem 2.1 in the case $m_1 > m_2$, we need to bound t^* by quantities that only depend on the initial datum.

Proof of Theorem 2.1 ($m_1 > m_2$). Let $u_0 \in L^1(\Omega) \cap L^{q_0}(\Omega) \cap L^\infty(\Omega)$. In the case $t^* = \infty$, from Corollary 4.2 we deduce the validity of

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0. \quad (4.22)$$

Since $m_1 > m_2$, it is straightforward to verify that (4.22) implies (2.1).

Suppose now that $t^* = 0$. This means that, by the definition of t^* , we can assume with no loss of generality that $\phi(u)$ is of porous medium type with $m = m_1$. In other words, we are allowed to replace (1.3)–(1.4) with

$$c_1 |u|^{m_1-1} \leq \phi'(u) \quad \forall u \in \mathbb{R}. \quad (4.23)$$

One can then proceed exactly as in the proofs of Lemma 4.1 and Corollary 4.2 (which in fact become simpler under (4.23)) to get

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0. \quad (4.24)$$

Again, the condition $m_1 > m_2$ ensures that from (4.24) there follows (2.1).

We are therefore left with the case $t^* \in (0, \infty)$. First of all note that, by taking t^* as the new time origin and reasoning as in the case $t^* = 0$ (by also using the non-expansivity of the norms), we obtain:

$$\|u(t)\|_\infty \leq K \left[(t - t^*)^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right] \quad \forall t > t^*. \quad (4.25)$$

Hence, by virtue of (4.17) and (4.25), we end up with

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0 + N(m_2 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_2 - 1)}} + \|u_0\|_{q_0} \right) & \forall t \in (0, t^*], \\ K \left[(t - t^*)^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right] & \forall t > t^*. \end{cases} \quad (4.26)$$

By the definition of t^* , and exploiting (4.26) evaluated at $t = t^*$, we can infer the inequality

$$1 \leq K \left(t^{*-\frac{N}{2q_0 + N(m_2 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_2 - 1)}} + \|u_0\|_{q_0} \right). \quad (4.27)$$

Let us first suppose

$$\|u_0\|_{q_0} \leq \frac{1}{2K}, \quad (4.28)$$

so that (4.27) yields

$$t^* \leq K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \quad K' := (2K)^{\frac{2q_0 + N(m_2 - 1)}{N}}. \quad (4.29)$$

On the other hand, the lower branch of (4.26) implies the validity of

$$\|u(t)\|_\infty \leq \underbrace{2^{\frac{N}{2q_0 + N(m_1 - 1)}} K}_{K''} \left(t^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right) \quad \forall t \geq 2t^*,$$

which in turn implies

$$\|u(t)\|_\infty \leq K'' \left(t^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right) \quad \forall t \geq 2K' \|u_0\|_{q_0}^{\frac{2q_0}{N}} \quad (4.30)$$

in view of (4.29). So, by exploiting the upper branch of (4.26), (4.30), recalling the definition of t^* and the non-expansivity of the norms, we deduce

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0 + N(m_2 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_2 - 1)}} + \|u_0\|_{q_0} \right) & \forall t \in (0, t^*], \\ 1 & \forall t \in \left(t^*, 2K' \|u_0\|_{q_0}^{\frac{2q_0}{N}} \right), \\ K'' \left(t^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right) & \forall t \geq 2K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}. \end{cases} \quad (4.31)$$

A straightforward computation shows that (4.31) is implied by

$$\|u(t)\|_\infty \leq \begin{cases} K''' \left(t^{-\frac{N}{2q_0 + N(m_2 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_2 - 1)}} + \|u_0\|_{q_0} \right) & \forall t \in \left(0, 2K' \|u_0\|_{q_0}^{\frac{2q_0}{N}} \right), \\ K'' \left(t^{-\frac{N}{2q_0 + N(m_1 - 1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0 + N(m_1 - 1)}} + \|u_0\|_{q_0} \right) & \forall t \geq 2K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \end{cases} \quad (4.32)$$

up to choosing

$$K''' := 2^{\frac{2q_0 + N(m_2 - 1)}{2q_0 + N(m_2 - 1)}} K.$$

It is then easy to check that from (4.32) estimate (2.1) follows provided one relabels the multiplicative constant K .

Let us now suppose that

$$\|u_0\|_{q_0} > \frac{1}{2K} \quad (4.33)$$

instead. In this case we can assume, with no loss of generality, the validity of

$$t^* > K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}. \quad (4.34)$$

Indeed, if (4.29) holds then the above argument leads to (4.32), since we only used (4.28) to get (4.29). Hence, under (4.34), we can easily deduce the analogue of (4.31):

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) & \forall t \in \left(0, K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}\right], \\ 1 & \forall t \in \left(K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, 2t^*\right), \\ K'' \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} + \|u_0\|_{q_0} \right) & \forall t \geq 2t^*. \end{cases} \quad (4.35)$$

In order to remove the presence of t^* in (4.35), let us show that the lower branch can be extended down to $t = K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}$. In fact, by evaluating the upper branch at such time and using the non-expansivity of the norms, we get:

$$\|u(t)\|_\infty \leq \frac{1}{2} + K \|u_0\|_{q_0} \quad \forall t \in \left(K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, 2t^*\right). \quad (4.36)$$

On the other hand, in view of (4.33), estimate (4.36) implies

$$\|u(t)\|_\infty \leq 2K \|u_0\|_{q_0} \quad \forall t \in \left(K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, 2t^*\right),$$

whence

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) & \forall t \in \left(0, K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}\right), \\ K''' \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} + \|u_0\|_{q_0} \right) & \forall t \geq K' \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \end{cases} \quad (4.37)$$

with $K''' := 2K \vee K''$. From (4.37) the validity of (2.1) is immediate upon relabelling the multiplicative constant K .

Finally, one can drop the assumption $u_0 \in L^\infty(\Omega)$ in a standard way since the right-hand side of estimate (2.1) does not depend on $\|u_0\|_\infty$, see for instance the proof of [29, Theorem 3.2]. \square

Let us now deal with the case $m_1 < m_2$, for which the analysis is much simpler because $\phi'(u)$ can be bounded from below by a single power.

Proof of Theorem 2.1 ($m_1 < m_2$). If m_1 is smaller than m_2 , it is plain that in view of (1.3)–(1.4) we have a double global lower bound on ϕ' :

$$(c_1 \wedge c_2) |u|^{m_1-1} \leq \phi'(u) \quad \forall u \in \mathbb{R}, \quad (4.38)$$

$$(c_1 \wedge c_2) |u|^{m_2-1} \leq \phi'(u) \quad \forall u \in \mathbb{R}. \quad (4.39)$$

This means that we can proceed exactly as in the proofs of Lemma 4.1 and Corollary 4.2 (with simplifications actually, since we can reason as if $m_1 = m_2 = m$) to get a double smoothing estimate, with no need for introducing t^* :

$$\begin{aligned} \|u(t)\|_\infty &\leq K \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0, \\ \|u(t)\|_\infty &\leq K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0. \end{aligned}$$

In particular there holds

$$\|u(t)\|_\infty \leq K \left[\left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \right) \wedge \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \right) + \|u_0\|_{q_0} \right] \quad (4.40)$$

for all $t > 0$, and it is immediate to check that (4.40) is equivalent to (2.1) up to a different constant K . \square

5. LONG-TIME ESTIMATES: PROOFS

In this section we shall complete our investigation by (mostly) addressing the long-time behaviour of solutions to (1.1), under the additional assumptions that Ω is of finite measure and that the Poincaré inequality (1.8) holds. To our ends the following proposition, which also has an independent interest, is crucial.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure, such that the Poincaré inequality (1.8) holds. For each $l \geq 1/2$, let Φ_l be any continuous function on \mathbb{R} satisfying*

$$(C_0 |y|)^l \leq |\Phi_l(y)| \leq (C_1 |y|)^l \quad \forall y \in \mathbb{R}, \quad (5.1)$$

$$y \Phi_l(y) > 0 \quad \forall y \neq 0, \quad (5.2)$$

for some positive constants $C_0 \leq C_1$ independent of l . Then there exists a positive constant C_P^* , depending only on Ω, C_1, C_2 , such that the nonlinear Poincaré inequality

$$\|\Phi_l(\xi)\|_2 \leq C_P^* \|\nabla \Phi_l(\xi)\|_2 \quad \forall \xi \in L^1(\Omega) : \Phi_l(\xi) \in H^1(\Omega), \quad \bar{\xi} = 0 \quad (5.3)$$

holds.

Proof. It is a consequence of the methods of proof of [29, Lemma 5.9] and [32, Lemma 5.6], hence we shall be concise and only point out the main differences (the present result is stated under more general assumptions).

If, by contradiction, the assertion is false, then there exist a sequence of numbers $\{l_n\} \subset [1/2, \infty)$ and a sequence of nontrivial functions $\{\xi_n\} \in L^1(\Omega)$ with $\Phi_{l_n}(\xi_n) \in H^1(\Omega)$, $\bar{\xi}_n = 0$, such that

$$\|\nabla \Phi_{l_n}(\xi_n)\|_2 \leq \frac{1}{n} \|\Phi_{l_n}(\xi_n)\|_2 \quad \forall n \in \mathbb{N}. \quad (5.4)$$

By setting $a_n := \|\Phi_{l_n}(\xi_n)\|_2$, $\Psi_n := \Phi_{l_n}(\xi_n)/a_n$ and applying the Poincaré inequality (1.8) to the function $f = \Psi_n$, together with (5.4), it is straightforward to deduce that the sequence $\{\Psi_n\}$ converges in $L^2(\Omega)$ to a constant $c_0 \neq 0$. We can assume with no loss of generality that $c_0 > 0$: in case $c_0 < 0$ one argues likewise in view of (5.2). Let us then set

$$\mathcal{Z}_n := a_n^{-\frac{1}{l_n}} \xi_n \quad \forall n \in \mathbb{N}.$$

Thanks to (5.1)–(5.2), there hold

$$|\mathcal{Z}_n| \leq \frac{1}{C_0} |\Psi_n|^{\frac{1}{l_n}} \quad \forall n \in \mathbb{N} \quad (5.5)$$

and

$$\liminf_{n \rightarrow \infty} \mathcal{Z}_n \geq \ell > 0, \quad \ell := \frac{\liminf_{n \rightarrow \infty} c_0^{\frac{1}{l_n}}}{C_1}. \quad (5.6)$$

In particular,

$$\lim_{n \rightarrow \infty} |E_n| = 0, \quad E_n := \{x \in \Omega : \mathcal{Z}_n(x) < 0\}. \quad (5.7)$$

Moreover, estimate (5.5) and Hölder's inequality yield

$$\int_{E_n} |\mathcal{Z}_n(x)| dx \leq \frac{1}{C_0} \int_{E_n} |\Psi_n(x)|^{\frac{1}{l_n}} dx \leq \frac{|E_n|^{1-\frac{1}{2l_n}}}{C_0} \|\Psi_n\|_{L^2(E_n)}^{\frac{1}{l_n}}. \quad (5.8)$$

Upon recalling that $\{l_n\} \subset [1/2, \infty)$ and that $\{\Psi_n\}$ converges in $L^2(\Omega)$, from (5.6)–(5.8) we infer

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{Z}_n(x) dx = \lim_{n \rightarrow \infty} \int_{\Omega \setminus E_n} \mathcal{Z}_n(x) dx \geq \ell |\Omega|,$$

a contradiction. □

As a direct consequence of Proposition 5.1 and the validity of the Gagliardo-Nirenberg-Sobolev inequalities (1.10), we have a further family of inequalities.

Corollary 5.2. *Let the hypotheses of Proposition 5.1 be fulfilled, and suppose in addition that the Gagliardo-Nirenberg-Sobolev inequalities (1.10) hold for all r, s complying with (1.12)–(1.13) and $\vartheta = \vartheta(s, r, N)$ as in (1.11). Then the nonlinear Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\Phi_l(\xi)\|_r \leq C_S^* \|\nabla \Phi_l(\xi)\|_2^\vartheta \|\Phi_l(\xi)\|_s^{1-\vartheta} \quad \forall \xi \in L^1(\Omega) : \Phi_l(\xi) \in H^1(\Omega) \cap L^s(\Omega), \quad \bar{\xi} = 0$$

hold for a positive constant C_S^* depending on Ω, C_1, C_2 and independent of $l \geq 1/2$ and r, s ranging in compact subsets of $(0, \infty)$.

Proof. It is enough to apply (1.10) to $f = \Phi_l(\xi)$ and combine it with (5.3):

$$\|\Phi_l(\xi)\|_r \leq C_S (\|\nabla \Phi_l(\xi)\|_2 + \|\Phi_l(\xi)\|_2)^\vartheta \|\Phi_l(\xi)\|_s^{1-\vartheta} \leq \underbrace{C_S (1 + C_P^*)^\vartheta}_{C_S^*} \|\nabla \Phi_l(\xi)\|_2^\vartheta \|\Phi_l(\xi)\|_s^{1-\vartheta}.$$

□

Following [29, Sections 4, 5], we now distinguish between data with *zero* mean and data with *nonzero* mean; note that the same property holds for the corresponding solutions to (1.1) thanks to *mass conservation* (Proposition 3.3).

5.1. The case $\bar{u}_0 = 0$: proof of Theorem 2.2. In order to prove Theorem 2.2, our first aim is to show that in the case of zero-mean data the corresponding solutions satisfy better smoothing effects (in fact analogous to those associated with the Dirichlet problem, see Remark 5.5 below). Afterwards, by borrowing some ideas firstly introduced in [10, 28] and then exploited in [29, 32], we prove *absolute bounds*, namely L^∞ estimates independent of the initial datum.

Lemma 5.3. *Let the hypotheses of Theorem 2.2 be fulfilled. Then the following smoothing estimate holds:*

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} & \forall t \in \left(0, \|u_0\|_{q_0}^{\frac{2q_0}{N}}\right), \\ K t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} & \forall t \geq \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \end{cases} \quad (5.9)$$

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, \Omega$.

Proof. Let p_k be any increasing sequence of positive numbers such that $p_0 = q_0$ and $p_\infty = \infty$. In view of the assumptions on Ω , we can apply Corollary 5.2 with the choices

$$\Phi_l(y) \equiv \Phi_k(y) = y^{\frac{m_i+p_k-1}{2}} \quad i = 1, 2;$$

moreover, since $\bar{u}_0 = 0$, mass conservation guarantees that $\bar{u}(t) = 0$ for all $t > 0$ as well. As a consequence, there holds

$$\frac{\|u(t)\|_{\frac{r(m_i+p_k-1)}{2}}^{\frac{m_i+p_k-1}{\vartheta}}}{C_S^{*\frac{2}{\vartheta}} \|u(t)\|_{\frac{s(m_i+p_k-1)}{2}}^{\frac{(1-\vartheta)(m_i+p_k-1)}{\vartheta}}} \leq \int_\Omega \left| \nabla \left(u^{\frac{m_i+p_k-1}{2}} \right) (x, t) \right|^2 dx \quad \forall t > 0, \quad i = 1, 2. \quad (5.10)$$

Hence, in the proof of Lemma 4.1 one can apply (5.10) to (4.4): as a result, a stronger version of inequality (4.5) holds (with C_S replaced by C_S^*), that is the same as above with no integral term in the right-hand side. All of the estimates carried out in Section 4 can then be recomputed by taking into account such an improvement: it is not difficult, though tedious, to check that the latter lead to the analogue of (2.1) with no additional term in the right-hand side, namely (5.9). □

Lemma 5.4. *Let the hypotheses of Theorem 2.2 be fulfilled. Then the following absolute bound holds:*

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{1}{m_2-1}} & \forall t \in (0, 1), \\ K t^{-\frac{1}{m_1-1}} & \forall t \geq 1, \end{cases} \quad (5.11)$$

where K is a positive constant depending only on $m_1, m_2, c_1, c_2, \Omega$.

Proof. Let us consider first the case $m_1 > m_2$. Let t^* be defined as in (4.1), and suppose as a first step that $t^* \in (0, \infty)$. With no loss of generality we can take again $u_0 \in L^\infty(\Omega)$ and set $M := \|u_0\|_\infty > 1$,

so that (4.3) holds. If we multiply the differential equation in (1.1) by u^{q_0-1} (let $q_0 > 1$), integrate in Ω and exploit (4.3), we obtain:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |u(x, t)|^{q_0} dx &= -q_0(q_0 - 1) \int_{\Omega} |u(x, t)|^{q_0-2} \phi'(u(x, t)) |\nabla u(x, t)|^2 dx \\ &\leq -\frac{c}{M^{m_1-m_2}} \int_{\Omega} |u(x, t)|^{m_1+q_0-3} |\nabla u(x, t)|^2 dx \\ &= -\frac{4c}{(m_1 + q_0 - 1)^2 M^{m_1-m_2}} \int_{\Omega} \left| \nabla \left(u^{\frac{m_1+q_0-1}{2}} \right) (x, t) \right|^2 dx. \end{aligned} \quad (5.12)$$

By arguing as in the proof of Lemma 5.3, in the r.h.s. of (5.12) we can apply the nonlinear Poincaré inequality (5.3) with the choice $\Phi_l(y) = y^{(m_1+q_0-1)/2}$, which yields

$$\frac{d}{dt} \int_{\Omega} |u(x, t)|^{q_0} dx \leq -\frac{4c}{|\Omega|^{\frac{m_1-1}{q_0}} C_P^{*2} (m_1 + q_0 - 1)^2 M^{m_1-m_2}} \left(\int_{\Omega} |u(x, t)|^{q_0} dx \right)^{\frac{m_1+q_0-1}{q_0}}; \quad (5.13)$$

by integrating (5.13) we end up with

$$\|u(t)\|_{q_0} \leq \left(\frac{C}{M^{m_1-m_2}} t + \|u_0\|_{q_0}^{1-m_1} \right)^{-\frac{1}{m_1-1}} \quad \forall t \geq 0, \quad (5.14)$$

where we set

$$C := \frac{4c q_0}{|\Omega|^{\frac{m_1-1}{q_0}} C_P^{*2} (m_1 - 1) (m_1 + q_0 - 1)^2}.$$

In particular, (5.14) implies

$$\|u(t)\|_{q_0} \leq C^{-\frac{1}{m_1-1}} \|u_0\|_{\infty}^{\frac{m_1-m_2}{m_1-1}} t^{-\frac{1}{m_1-1}} \quad \forall t \geq 0. \quad (5.15)$$

Now we notice that, by means of the same methods of proof as in Section 4, the upper branch of estimate (5.9) in fact holds up to $t = t^*$. As a consequence, thanks to a $t/2$ -shift argument, we infer that

$$\|u(t)\|_{\infty} \leq 2^{\frac{N}{2q_0+N(m_2-1)}} K t^{-\frac{N}{2q_0+N(m_2-1)}} \|u(t/2)\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \quad \forall t \in (0, t^*).$$

On the other hand, by evaluating (5.15) at $t/2$ and plugging it into (5.9) we obtain

$$\|u(t)\|_{\infty} \leq C' t^{-\frac{2q_0+N(m_1-1)}{(m_1-1)[2q_0+N(m_2-1)]}} \|u_0\|_{\infty}^{\frac{2q_0(m_1-m_2)}{(m_1-1)[2q_0+N(m_2-1)]}} \quad \forall t \in (0, t^*), \quad (5.16)$$

where $C' > 0$ is a suitable constant depending on m_1, m_2, N, q_0, K, C . It is straightforward to check that a routine iteration of (5.16) (which still exploits a $t/2$ -shift argument) yields

$$\|u(t)\|_{\infty} \leq K t^{-\frac{1}{m_2-1}} \quad \forall t \in (0, t^*) \quad (5.17)$$

for some $K > 0$ as in the statement (the role of q_0 here is inessential), which will not be relabelled from here on. For $t > t^*$ one can reason exactly as if $\phi(u)$ is of porous medium type with $m = m_1$ (recall the beginning of the proof of Theorem 2.1, case $m_1 > m_2$). This gives rise to the analogue of (5.17), namely

$$\|u(t)\|_{\infty} \leq K (t - t^*)^{-\frac{1}{m_1-1}} \quad \forall t > t^*. \quad (5.18)$$

Thanks to (5.17), by the definition of t^* , we can therefore deduce that

$$t^* \leq T := K^{m_2-1}. \quad (5.19)$$

In particular, by combining (5.18) with (5.19) we get

$$\|u(t)\|_{\infty} \leq K t^{-\frac{1}{m_1-1}} \quad \forall t > 2T; \quad (5.20)$$

hence, upon collecting (5.17), (5.19), (5.20) and the non-expansivity of the norms, we end up with

$$\|u(t)\|_{\infty} \leq \begin{cases} K t^{-\frac{1}{m_2-1}} & \forall t \in (0, t^*], \\ 1 & \forall t \in (t^*, 2T), \\ K t^{-\frac{1}{m_1-1}} & \forall t \geq 2T. \end{cases} \quad (5.21)$$

By arguing as in the proof of Theorem 2.1 (case $m_1 > m_2$), up to choosing a larger constant K it is apparent that estimate (5.21) is implied by

$$\|u(t)\|_{\infty} \leq \begin{cases} K t^{-\frac{1}{m_2-1}} & \forall t \in (0, 2T), \\ K t^{-\frac{1}{m_1-1}} & \forall t \geq 2T, \end{cases}$$

whence (5.11).

In the case where $t^* = 0$, by arguing as in previous computations it is direct to see that (5.20) holds for all $t > 0$: since $m_1 > m_2$, such an estimate trivially implies (5.11). In the case where $t^* = \infty$, clearly (5.17) holds for all $t > 0$, from which again (5.11) follows.

Let us finally discuss the case $m_1 < m_2$: as remarked in the corresponding proof of Theorem 2.1, both the lower bound (4.38) and (4.39) on ϕ' hold. In particular, by reasoning as in the proof of Lemma 5.3, we can deduce the validity of both the estimate

$$\|u(t)\|_\infty \leq K t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \quad \forall t > 0 \quad (5.22)$$

and

$$\|u(t)\|_\infty \leq K t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \quad \forall t > 0. \quad (5.23)$$

Moreover, by exploiting again (4.38)–(4.39) and arguing similarly to the first part of the proof (with simplifications since we are basically in the single-power case), we obtain the L^{q_0} -absolute bounds

$$\|u(t)\|_{q_0} \leq K t^{-\frac{1}{m_1-1}} \quad \text{and} \quad \|u(t)\|_{q_0} \leq K t^{-\frac{1}{m_2-1}} \quad \forall t > 0. \quad (5.24)$$

By gathering (5.22)–(5.24) through one $t/2$ -shift step, we deduce the L^∞ -absolute bounds

$$\|u(t)\|_\infty \leq K t^{-\frac{1}{m_1-1}} \quad \text{and} \quad \|u(t)\|_\infty \leq K t^{-\frac{1}{m_2-1}} \quad \forall t > 0. \quad (5.25)$$

It is then straightforward to check that (5.25) is equivalent to (5.11), since $m_1 < m_2$. \square

We are now in position to prove Theorem 2.2.

Proof of Theorem 2.2. Let us start again from the case $m_1 > m_2$. If we combine estimate (5.9) with (5.11), we obtain:

$$\|u(t)\|_\infty \leq \begin{cases} K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \wedge t^{-\frac{1}{m_2-1}} \right) & \forall t \in \left(0, \|u_0\|_{q_0}^{\frac{2q_0}{N}} \wedge 1 \right), \\ K \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \wedge t^{-\frac{1}{m_1-1}} \right) & \forall t > \|u_0\|_{q_0}^{\frac{2q_0}{N}} \vee 1. \end{cases} \quad (5.26)$$

Let us first deal with the case $\|u_0\|_{q_0} \leq 1$. Up to a different multiplicative constant K , under such assumption it is not difficult to check that (5.26) is equivalent to

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} & \forall t \in \left(0, \|u_0\|_{q_0}^{\frac{2q_0}{N}} \right), \\ K t^{-\frac{N}{2q_0+N(m_1-1)}} (t + \|u_0\|_{q_0}^{1-m_1})^{-\frac{2q_0}{(m_1-1)[2q_0+N(m_1-1)]}} & \forall t > 1. \end{cases} \quad (5.27)$$

As concerns the upper branch, it is enough to compare the two time powers involved in the minimum in (5.26) both as $t \downarrow 0$ and at $t = \|u_0\|_{q_0}^{2q_0/N}$: one sees that the first one is always smaller. On the other hand, by means of the change of variables $\tau = \|u_0\|_{q_0}^{m_1-1} t$, one can show that the lower branches of (5.26) and (5.27) are indeed equivalent. We are therefore left with providing an estimate in the region

$$\|u_0\|_{q_0}^{\frac{2q_0}{N}} \leq t \leq 1.$$

Here we have to exploit the lower branch of (5.9) and the upper branch of (5.11), which yield

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \wedge t^{-\frac{1}{m_2-1}} \right) \quad \forall t \in \left[\|u_0\|_{q_0}^{\frac{2q_0}{N}}, 1 \right]. \quad (5.28)$$

By comparing the two time powers of (5.28) at the extremals $t = \|u_0\|_{q_0}^{2q_0/N}$ and $t = 1$, it is straightforward to show that the first one is always smaller, so that (5.28) is in fact equivalent to

$$\|u(t)\|_\infty \leq K t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \quad \forall t \in \left[\|u_0\|_{q_0}^{\frac{2q_0}{N}}, 1 \right]. \quad (5.29)$$

Clearly, (5.27) and (5.29) give (2.2).

We finally deal with the case $\|u_0\|_{q_0} > 1$. By reasoning in a similar way to above, one can check that (5.26) is now the same as

$$\|u(t)\|_\infty \leq \begin{cases} K t^{-\frac{N}{2q_0+N(m_2-1)}} (t + \|u_0\|_{q_0}^{1-m_2})^{-\frac{2q_0}{(m_2-1)[2q_0+N(m_2-1)]}} & \forall t \in (0, 1), \\ K t^{-\frac{1}{m_1-1}} & \forall t > \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \end{cases} \quad (5.30)$$

up to a different constant K . In the intermediate region

$$1 \leq t \leq \|u_0\|_{q_0}^{\frac{2q_0}{N}},$$

by combining the upper branch of (5.9) with the lower branch of (5.11) we get

$$\|u(t)\|_\infty \leq K \left(t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \wedge t^{-\frac{1}{m_1-1}} \right) \quad \forall t \in \left[1, \|u_0\|_{q_0}^{\frac{2q_0}{N}} \right]. \quad (5.31)$$

By comparing the two time powers of (5.31) at the extremals $t = 1$ and $t = \|u_0\|_{q_0}^{2q_0/N}$, it is direct to show that the latter is equivalent to

$$\|u(t)\|_\infty \leq K t^{-\frac{1}{m_1-1}} \quad \forall t \in \left[1, \|u_0\|_{q_0}^{\frac{2q_0}{N}} \right],$$

which, together with (5.30), gives rise to (2.3).

In order to complete the proof, we are left with addressing the case $m_1 < m_2$. Actually in the above computations we never used the fact that $m_1 > m_2$, so they also hold for $m_1 < m_2$. Hence, we just need to make sure that (2.2)–(2.3) are not worse than

$$t^{-\frac{N}{2q_0+N(m_2-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_2-1)}} \wedge t^{-\frac{1}{m_2-1}} \wedge t^{-\frac{N}{2q_0+N(m_1-1)}} \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m_1-1)}} \wedge t^{-\frac{1}{m_1-1}}, \quad (5.32)$$

namely the best estimate one can get from Lemmas 5.3–5.4 (recall in particular the end of proof of Lemma 5.4). To this end, because we only deal with time-power functions, in the case $\|u_0\|_{q_0} \leq 1$ it is enough to compare (2.2) and (5.32) at

$$t \downarrow 0, \quad t = \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \quad t = 1, \quad t = \|u_0\|_{q_0}^{1-m_1}, \quad t = \|u_0\|_{q_0}^{1-m_2}, \quad t \rightarrow \infty,$$

whereas in the case $\|u_0\|_{q_0} > 1$ it is enough to compare (2.3) and (5.32) at

$$t \downarrow 0, \quad t = \|u_0\|_{q_0}^{1-m_2}, \quad t = \|u_0\|_{q_0}^{1-m_1}, \quad t = 1, \quad t = \|u_0\|_{q_0}^{\frac{2q_0}{N}}, \quad t \rightarrow \infty.$$

A straightforward check yields the assertion. \square

Remark 5.5. As mentioned previously, the same smoothing effects as in Lemma 5.3 also hold for solutions of the *homogeneous Dirichlet* problem

$$\begin{cases} u_t = \Delta \phi(u) & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

Furthermore, in such case Ω can be *any* domain of \mathbb{R}^N : this is due to the fact that the Gagliardo-Nirenberg-Sobolev inequalities (let r, s, ϑ be as in (1.11)–(1.13))

$$\|f\|_r \leq C_S \|\nabla f\|_2^{\vartheta(s,r,N)} \|f\|_s^{1-\vartheta(s,r,N)}$$

hold for all $f \in \mathcal{D}(\mathbb{R}^N)$, hence in the whole $H_0^1(\Omega) \cap L^s(\Omega)$ (or even $\dot{H}^1 \cap L^s(\Omega)$) by density. As for the improved estimates of Theorem 2.2, one should require the finiteness of the measure of Ω , or more generally the validity of a sub-Poincaré inequality according to [31, Theorem 4.1], from which absolute bounds follow.

5.2. The case $\bar{u}_0 \neq 0$: proof of Theorem 2.3. If the mean value of the initial datum is not zero, according to [29, Section 5], the first step in order to understand the long-time behaviour of the solution is the proof of the uniform convergence to the mean value itself (which, we recall, is preserved in view of Proposition 3.3).

Lemma 5.6. *Let the hypotheses of Theorem 2.1 be fulfilled. Suppose moreover that Ω is of finite measure and such that the Poincaré inequality (1.8) holds. Let $\bar{u}_0 \neq 0$. Then the following smoothing estimate holds:*

$$\|u(t) - \bar{u}_0\|_\infty \leq K_0 t^{-\frac{N}{2}} \|u_0 - \bar{u}_0\|_1 \quad \forall t \geq 1, \quad (5.33)$$

where K_0 is a positive constant depending on $\|u_0\|_1, |\bar{u}_0|, m_1, m_2, c_1, c_2, \Omega$, which can be assumed to be increasing w.r.t. $\|u_0\|_1$ and locally bounded w.r.t. $|\bar{u}_0| > 0$.

Proof. We proceed along the lines of proof of [29, Theorem 5.10], hence we do not give full technical details. As usual we take $u_0 \in L^\infty(\Omega)$, with no loss of generality. By virtue of the mass-conservation property ensured by Proposition 3.3, we know that $\bar{u}(t) = \bar{u}_0$ for all $t > 0$. Hence, if we let e.g. $p \geq 2$, multiply (1.1) by $(u(t) - \bar{u}_0)^{p-1}$ and integrate by parts in $\Omega \times (t_1, t_2)$, we (formally) obtain:

$$\|u(t_2) - \bar{u}_0\|_p^p + p(p-1) \int_{t_1}^{t_2} \int_{\Omega} |u(x, t) - \bar{u}_0|^{p-2} \phi'(u(x, t)) |\nabla u(x, t)|^2 dx dt = \|u(t_1) - \bar{u}_0\|_p^p \quad (5.34)$$

for all $t_2 > t_1 > 0$. As a consequence, we also deduce that

$$\|u(t_2) - \bar{u}_0\|_p \leq \|u(t_1) - \bar{u}_0\|_p \quad \forall p \in [1, \infty], \quad \forall t_2 > t_1 \geq 0; \quad (5.35)$$

we shall refer to such property as *non-expansivity of the differences* (in $L^p(\Omega)$ between the solution and its mean value, or any constant actually), in agreement with (3.4). Note that in fact (5.35) holds for all $p \in [1, \infty]$. The justification of the above computations follows by remarks similar to those pointed out in the proof of Lemma 4.1. Thanks to Theorem 2.1 (with $q_0 = 1$) and conditions (1.3)–(1.4), we have that

$$\tilde{c} |u(x, t)|^{m_1-1} \leq \phi'(u(x, t)) \quad \text{for a.e. } (x, t) \in \Omega \times [1/2, \infty), \quad (5.36)$$

for a suitable $\tilde{c} > 0$ depending on $m_1, m_2, c_1, c_2, C_S, N, \|u_0\|_1$. Estimate (5.36) combined with (5.34) yields

$$\|u(t_2) - \bar{u}_0\|_p^p + \tilde{c} p(p-1) \int_{t_1}^{t_2} \int_{\Omega} |u(x, t) - \bar{u}_0|^{p-2} |u(x, t)|^{m_1-1} |\nabla u(x, t)|^2 dx dt \leq \|u(t_1) - \bar{u}_0\|_p^p \quad (5.37)$$

for all $t_2 > t_1 > 1/2$. Upon introducing the *relative error* $w := u/\bar{u}_0 - 1$, inequality (5.37) reads

$$\|w(t_2)\|_p^p + \tilde{c} |\bar{u}_0|^{m_1-1} p(p-1) \int_{t_1}^{t_2} \int_{\Omega} |\nabla \Phi_p(w)(x, t)|^2 dx dt \leq \|w(t_1)\|_p^p, \quad (5.38)$$

where we set

$$\Phi_p(y) := \int_0^y |r|^{\frac{p}{2}-1} |r+1|^{\frac{m_1-1}{2}} dr \quad \forall y \in \mathbb{R}.$$

In Lemma 5.18 of [29] it was shown that Φ_p satisfies

$$\frac{\tilde{C}}{p^{1+1\vee\frac{m_1-1}{2}}} |y|^{\frac{p}{2}} \leq |\Phi_p(y)| \leq \frac{(R+1)^{\frac{m_1-1}{2}}}{p} |y|^{\frac{p}{2}} \quad \forall y \in [-R, R], \quad \forall R > 1, \quad (5.39)$$

for a suitable positive constant \tilde{C} depending only on m_1 . It is direct to check that Φ_p can be modified in $(-\infty, -R) \cup (R, \infty)$ so as to satisfy (5.39) in the whole \mathbb{R} : we denote by Φ_p^R such function. Still as a consequence of Theorem 2.1, we know that there exists $R = R_0 > 1$, depending only on the quantities $\|u_0\|_1, |\bar{u}_0|, m_1, m_2, c_1, c_2, C_S, N$, such that

$$|w(x, t)| \leq \frac{\|u(t) - \bar{u}_0\|_\infty}{|\bar{u}_0|} \leq R_0 \quad \text{for a.e. } (x, t) \in \Omega \times [1/2, \infty).$$

Hence, (5.38) is equivalent to

$$\|w(t_2)\|_p^p + \tilde{c} |\bar{u}_0|^{m_1-1} p(p-1) \int_{t_1}^{t_2} \int_{\Omega} |\nabla \Phi_p^{R_0}(w)(x, t)|^2 dx dt \leq \|w(t_1)\|_p^p. \quad (5.40)$$

We are therefore in position to apply Corollary 5.2 to $\Phi_l \equiv \Phi_p^{R_0}$ in (5.40), which yields

$$\|w(t_2)\|_p^p + D p(p-1) \int_{t_1}^{t_2} \frac{\|\Phi_p^{R_0}(w(t))\|_r^{\frac{2}{\vartheta}}}{\|\Phi_p^{R_0}(w(t))\|_s^{\frac{2(1-\vartheta)}{\vartheta}}} dt \leq \|w(t_1)\|_p^p, \quad (5.41)$$

where from here on D stands for a generic positive constant that can be taken to be depending only on $\|u_0\|_1, |\bar{u}_0|, m_1, m_2, c_1, c_2, \Omega$, whereas $r, s \geq 2$ will be chosen below. By exploiting (5.39) (with $R = R_0$) and the non-expansivity of the differences, from (5.41) we infer

$$\frac{D \tilde{C}^{\frac{2}{\vartheta}} (p-1)}{p^{\frac{2}{\vartheta}(\vartheta+1\vee\frac{m_1-1}{2})-1} (R_0+1)^{\frac{(1-\vartheta)(m_1-1)}{\vartheta}}} (t_2 - t_1) \frac{\|w(t_2)\|_{rp}^{\frac{p}{2}}}{\|w(t_1)\|_{\frac{rp}{s}}^{\frac{p(1-\vartheta)}{2}}} \leq \|w(t_1)\|_p^p.$$

After some algebra, it is straightforward to check that the choices $s = 2$ and $r = 2 + 4/N$ entail

$$\|w(t_2)\|_{\frac{N+2}{N}p} \leq D^{\frac{\log p}{p}} (t_2 - t_1)^{-\frac{N}{(N+2)p}} \|w(t_1)\|_p. \quad (5.42)$$

For any given $t > 1/2$, let us replace t_1 with t_n , t_2 with t_{n+1} , p with p_n and set

$$t_n := \frac{1}{2} + \left(1 - \frac{1}{2^n}\right) \left(t - \frac{1}{2}\right), \quad p_n := 2 \left(\frac{N+2}{N}\right)^n \quad \forall n \in \mathbb{N},$$

so that by iterating (5.42) similarly to the proof of [29, Theorem 5.10] (using again the non-expansivity of the differences), one ends up with

$$\|w(t)\|_\infty \leq D (t - 1/2)^{-\frac{N}{4}} \|w(1/2)\|_2 \leq D t^{-\frac{N}{4}} \|w_0\|_2 \quad \forall t \geq 1.$$

Estimate (5.33) then just follows by standard interpolation plus an iterative $t/2$ -shift argument. The fact that the multiplicative constant can be taken to be increasing w.r.t. $\|u_0\|_1$ is just a consequence of the method of proof, since the same property holds for the r.h.s. of (2.1) and for R_0 . As concerns the dependence on $|\bar{u}_0|$, analogous remarks apply. \square

We can finally establish *exponential* uniform convergence to the mean value, in contrast with the zero-mean case.

Proof of Theorem 2.3. We follow closely the strategy used in the proofs of [29, Theorems 4.3 and 5.11]. First of all, since (5.33) yields uniform convergence to the mean value by means of a quantitative estimate, we deduce that there exists $t_0 \geq 1$, depending on $\|u_0\|_1, |\bar{u}_0|, m_1, m_2, c_1, c_2, \Omega$, such that

$$|u(x, t)| \geq \frac{|\bar{u}_0|}{2} \quad \text{for a.e. } (x, t) \in \Omega \times (t_0, \infty). \quad (5.43)$$

Upon taking advantage of (5.43) in (5.37) (in the case $p = 2$), the Poincaré inequality (1.8) and reasoning exactly as in the proof [29, Theorem 4.3], we end up with

$$\|u(t) - \bar{u}_0\|_2 \leq e^{-\frac{\varepsilon |\bar{u}_0|^{m_1-1}}{2^{m_1-1} c_P^2} (t-t_0)} \|u(t_0) - \bar{u}_0\|_2 \quad \forall t \geq t_0. \quad (5.44)$$

If we exploit (5.33) between t and $t - 1$, use (5.44) and the smoothing effect (2.1), we obtain:

$$\|u(t) - \bar{u}_0\|_\infty \leq K_0 e^{-\frac{\varepsilon |\bar{u}_0|^{m_1-1}}{2^{m_1-1} c_P^2} t} \quad \forall t \geq t_0 + 1, \quad (5.45)$$

for another positive constant K_0 depending on the same quantities as the one in (5.33). In view of (5.45) and the fact that ϕ is C^2 in a neighbourhood of \bar{u}_0 , we can infer that there exist positive constants $t_1 = t_1(\|u_0\|, \bar{u}_0, \phi, \Omega)$, $K_1 = K_1(\|u_0\|, \bar{u}_0, \phi, \Omega)$ and $M = M(|\bar{u}_0|, m_1, \tilde{c}, C_P)$ such that

$$\phi'(u(x, t)) \geq \phi'(\bar{u}_0) - K_1 e^{-Mt} \quad \forall t \geq t_1. \quad (5.46)$$

By plugging (5.46) in (5.34) (with $p = 2$) and carrying out similar computations to those performed in the proof of [29, Theorem 4.3], we infer the inequality

$$\|u(t) - \bar{u}_0\|_2 \leq e^{-\frac{1}{c_P^2} \int_{t_1}^t [\phi'(\bar{u}_0) - K_1 e^{-Ms}] ds} \|u(t_1) - \bar{u}_0\|_2 \quad \forall t \geq t_1. \quad (5.47)$$

It is then apparent that (2.4) is a consequence of (5.33) applied between t and $t - 1$, (5.47), again (5.33) and the non-expansivity of the differences.

As for the dependence of the multiplicative constant on $\|u_0\|_1$ and $|\bar{u}_0|$, the same comments as in the end of the proof of Lemma 5.6 hold. \square

Remark 5.7. For simplicity, in Theorem 2.3 we assumed that ϕ is C^2 away from 0. However, as it can be guessed from the above proof, the result continues to hold under the milder hypothesis

$$\int_0^1 \frac{\omega(r)}{r} dr < \infty,$$

where $\omega : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is the local modulus of continuity of ϕ' in $\mathbb{R} \setminus \{0\}$.

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REFERENCES

- [1] R. A. Adams, J. J. F. Fournier, “Sobolev Spaces. Second Edition”, Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.
- [2] N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations 4 (1979), 827–868.
- [3] N. D. Alikakos, R. Rostamian, Large time behavior of solutions of Neumann boundary value problem for the porous medium equation, Indiana Univ. Math. J. 30 (1981), 749–785.
- [4] D. Andreucci, G. R. Cirmi, S. Leonardi, A. F. Tedeev, Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary, J. Differential Equations 174 (2001), 253–288.
- [5] D. Andreucci, A. F. Tedeev, Sharp estimates and finite speed of propagation for a Neumann problem in domains narrowing at infinity, Adv. Differential Equations 5 (2000), 833–860.
- [6] D. Andreucci, A. F. Tedeev, The Cauchy-Dirichlet problem for the porous media equation in cone-like domains, SIAM J. Math. Anal. 46 (2014), 1427–1455.
- [7] D. Andreucci, A. F. Tedeev, Optimal decay rate for degenerate parabolic equations on noncompact manifolds, Methods Appl. Anal. 22 (2015), 359–376.
- [8] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44 (1995), 1033–1074.
- [9] M. Bonforte, F. Cipriani, G. Grillo, Ultracontractivity and convergence to equilibrium for supercritical quasilinear parabolic equations on Riemannian manifolds, Adv. Differential Equations 8 (2003), 843–872.
- [10] M. Bonforte, G. Grillo, Asymptotics of the porous media equation via Sobolev inequalities, J. Funct. Anal. 225 (2005), 33–62.
- [11] M. Bonforte, G. Grillo, Super and ultracontractive bounds for doubly nonlinear evolution equations, Rev. Mat. Iberoam. 22 (2006), 111–129.
- [12] M. Bonforte, G. Grillo, J. L. Vázquez, Fast diffusion flow on manifolds of nonpositive curvature, J. Evol. Equ. 8 (2008), 99–128.
- [13] M. Bonforte, J. L. Vázquez, A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains, Arch. Ration. Mech. Anal. 218 (2015), 317–362.
- [14] M. Bonforte, J. L. Vázquez, Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds, Nonlinear Anal. 131 (2016), 363–398.
- [15] T. Coulhon, D. Hauer, Regularisation effects of nonlinear semigroups, preprint arXiv: <http://arxiv.org/abs/1604.08737>.
- [16] P. Daskalopoulos, C. E. Kenig, “Degenerate diffusions. Initial value problems and local regularity theory”, EMS Tracts in Mathematics, 1. European Mathematical Society (EMS), Zürich, 2007.
- [17] E. B. Davies, “Heat Kernels and Spectral Theory”, Cambridge Tracts in Mathematics, 92, Cambridge University Press, Cambridge, 1989.
- [18] F. del Teso, J. Endal, E. R. Jakobsen, On the well-posedness of solutions with finite energy for nonlocal equations of porous medium type, preprint arXiv: <https://arxiv.org/abs/1610.02221>.
- [19] A. de Pablo, F. Quirós, A. Rodríguez, Nonlocal filtration equations with rough kernels, Nonlinear Anal. 137 (2016), 402–425.
- [20] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A general fractional porous medium equation, Comm. Pure Appl. Math. 45 (2012), 1242–1284.
- [21] J. Dolbeault, I. Gentil, A. Guillin and F.-Y. Wang, L^q -functional inequalities and weighted porous media equations, Potential Anal. 28 (2008), 35–59.
- [22] S. D. Eidelman, S. Kamin, On stabilization of solutions of the Cauchy problem for parabolic equations degenerating at infinity, Asymptot. Anal. 45 (2005), 55–71.
- [23] S. D. Eidelman, S. Kamin, A. F. Tedeev, On stabilization of solutions of the Cauchy problem for linear degenerate parabolic equations, Adv. Differential Equations 14 (2009), 621–641.
- [24] D. Eidus, The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium, J. Differential Equations 84 (1990), 309–318.
- [25] D. Eidus, S. Kamin, The filtration equation in a class of functions decreasing at infinity, Proc. Amer. Math. Soc. 120 (1994), 825–830.
- [26] S. Kamin, P. Rosenau, Nonlinear diffusion in a finite mass medium, Comm. Pure Appl. Math. 35 (1982), 113–127.
- [27] E. Gagliardo, Proprietà di alcune classi di funzioni in più variabili (Italian), Ricerche Mat. 7 (1958), 102–137.
- [28] G. Grillo, On the equivalence between p -Poincaré inequalities and L^r - L^q regularization and decay estimates of certain nonlinear evolutions, J. Differential Equations 249 (2010), 2561–2576.
- [29] G. Grillo, M. Muratori, Sharp short and long time L^∞ bounds for solutions to porous media equations with homogeneous Neumann boundary conditions, J. Differential Equations 254 (2013), 2261–2288.
- [30] G. Grillo, M. Muratori, Sharp asymptotics for the porous media equation in low dimensions via Gagliardo-Nirenberg inequalities, Riv. Math. Univ. Parma (N.S.) 5 (2014), 15–38.
- [31] G. Grillo, M. Muratori, Smoothing effects for the porous medium equation on Cartan-Hadamard manifolds, Nonlinear Anal. 131 (2016), 346–362.
- [32] G. Grillo, M. Muratori, M. M. Porzio, Porous media equations with two weights: smoothing and decay properties of energy solutions via Poincaré inequalities, Discrete Contin. Dyn. Syst. 33 (2013), 3599–3640.
- [33] G. Grillo, M. Muratori, F. Punzo, Conditions at infinity for the inhomogeneous filtration equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 413–428.
- [34] G. Grillo, M. Muratori, F. Punzo, Fractional porous media equations: existence and uniqueness of weak solutions with measure data, Calc. Var. Partial Differential Equations 54 (2015), 3303–3335.

- [35] M. Guedda, D. Hilhorst, M. A. Peletier, *Disappearing interfaces in nonlinear diffusion*, Adv. Math. Sci. Appl. 7 (1997), 695–710.
- [36] E. Hebey, “Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities”, Courant Lecture Notes in Mathematics, 5, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
- [37] S. Kamin, *Similar solutions and the asymptotics of filtration equations*, Arch. Rational Mech. Anal. 60 (1975/76), 171–183.
- [38] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. 17 (1964), 101–134.
- [39] J. Moser, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math. 24 (1971), 727–740.
- [40] M. Muratori, “Weighted Functional Inequalities and Nonlinear Diffusions of Porous Medium Type”, Ph.D. Thesis, Politecnico di Milano and Université Paris 1 Panthéon-Sorbonne (2015), available online at <https://hal.archives-ouvertes.fr/tel-01289874>.
- [41] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa 13 (1959), 115–162.
- [42] O. A. Oleĭnik, A. S. Kalašnikov, Y.-L. Čžou, *The Cauchy problem and boundary problems for equations of the type of non-stationary filtration* (Russian), Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 667–704.
- [43] M. M. Porzio, *On decay estimates*, J. Evol. Equ. 9 (2009), 561–591.
- [44] M. M. Porzio, *Existence, uniqueness and behavior of solutions for a class of nonlinear parabolic problems*, Nonlinear Anal. 74 (2011), 5359–5382.
- [45] M. M. Porzio, *On uniform and decay estimates for unbounded solutions of partial differential equations*, J. Differential Equations 259 (2015), 6960–7011.
- [46] T. A. Sanikidze, A. F. Tedeev, *On the temporal decay estimates for the degenerate parabolic system*, Commun. Pure Appl. Anal. 12 (2013), 1755–1768.
- [47] A. F. Tedeev, V. Vespri, *Optimal behavior of the support of the solutions to a class of degenerate parabolic systems*, Interfaces Free Bound. 17 (2015), 143–156.
- [48] J. L. Vázquez, “Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type”, Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.
- [49] J. L. Vázquez, “The Porous Medium Equation. Mathematical Theory”, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.
- [50] J. L. Vázquez, A. de Pablo, F. Quirós, A. Rodríguez, *Classical solutions and higher regularity for nonlinear fractional diffusion equations*, to appear on J. Eur. Math. Soc., preprint arXiv: <http://arxiv.org/abs/1311.7427>.

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